chengitter-Struktur ebenfalls nicht festzustellen (bezüglich Hg vgl. die Pfeile ↓ in den Abb. 1 a und b). Auch hier ist die Übereinstimmung der beobachteten Atomabstände mit den berechneten Abstandswerten überraschend gut.

Die umfangreichen FOURIER-Analysen wurden mittels des elektronischen Rechenautomaten (ER 56) wieder von Herrn Dipl.-Phys. R. LEONHARDT durchgeführt. Wir danken ihm hierfür herzlich, ebenso danken wir der Deutschen Forschungsgemeinschaft für die vielseitige Unterstützung der vorliegenden Untersuchungen.

Nonlinear Incoherent Light Scattering

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(Z. Naturforschg. **22 a**, 662—670 [1967]; received 3 February 1967)

Classical equations are used to calculate nonlinear incoherent light scattered from two coherent beams of electromagnetic radiation of different frequencies propagating in a homogeneous plasma without magnetic field. With a suitable choice of the difference frequency, enhanced light scattering occurs near the plasma frequency. In this case the frequency spectrum is mainly given by the thermal ion density fluctuations.

Nonlinear processes, occurring when one or more electromagnetic waves propagate in a collisionless plasma, have been examined both theoretically [1–8] and experimentally [9]. Theoretical work is of two kinds. In the first type, the plasma is treated using a collective model, be it the VLASOV equation or macroscopic equations. This produces nonlinear waves with frequencies, wave numbers and directions uniquely determined by the incoming (primary) waves [4–9] because of energy and momentum conservation. (Though, as a result of refraction, a finite plasma volume makes a finite width of the angle of propagation.) In the second type [1–3] the discrete plasma structure is considered explicitly, which leads to new effects. Each particle separately can exchange energy and momentum with the waves. Since, moreover, all particles interact via COULOMB forces, this presents an intricate problem which to the author's knowledge has been treated hitherto mainly in terms of quantum field theory with results of various authors that are not in agreement [2]. This work contains a purely classical description of nonlinear wave phenomena in which the particle aspects of the plasma are included, and the difficulties involved in quantum field theoretical representations are avoided.

In section 1 the problem is specified and the equations for handling it are given. Sections 2 and 3 give a brief summary of results relating to thermal plasmas and linear wave effects. In section 4 nonlinear effects are derived, which are discussed in sections 5 and 6.

1. Basic Equations

The nonlinear interaction of electromagnetic waves will be specified here in the following way: Two electromagnetic waves having the frequencies ω_σ, $\sigma = 1, 2$ the directions $\boldsymbol{n}_\sigma$ and arbitrary polarisation propagate in a homogeneous isotropic plasma: What are the nonlinear (quadratic) effects that can be seen outside the plasma?

1 H. CHENG and Y. C. LEE, Phys. Rev. **142**, 104 [1966].
2 G. BAYM and R. W. HELLWARTH, IEEE J. Quant. Electronics QE-1, 309 [1965].
3 P. M. PLATZMANN and N. TZOAR, Phys. Rev. **136**, 11 [1964]. — D. F. DUBOIS and V. GILINSKY, Phys. Rev. **135**, 995 [1964]. — H. L. BERK, Phys. Fluids **7**, 917 [1964].
4 H. S. C. WANG and M. S. LAJKO, Phys. Fluids **6**, 1458 [1963].
5 W. H. KEGEL, Z. Naturforschg. **20 a**, 793 [1965]. — D. MONTGOMERY, Physica **31**, 789 [1965].
6 N. M. KROLL, A. RON, and N. ROSTOKER, Phys. Rev. Letters **13**, 83 [1964]. — A. SALAT, Z. Naturforschg. **20 a**, 690 [1965].
7 R. F. WHITMER and E. B. BARRET, Phys. Rev. **121**, 661 [1961]; **125**, 1478 [1962]. — L. M. GORBUNOV, V. V. PUSTOVALOV, and V. P. SILIN, Soviet Phys.-JETP **20**, 967 [1965]. — A. SALAT and A. SCHLÜTER, Z. Naturforschg. **20 a**, 458 [1965].
8 R. F. WHITMER, E. B. BARRET, and S. J. TETENBAUM, Phys. Rev. **135**, 369 [1964].
9 B. BLACHIER, J.-L. DELCROIX, and E. LEIBA, C. R. Acad. Sci. Paris **262**, 472 [1966]. — R. F. WHITMER, E. B. BARRET, and S. J. TETENBAUM, Phys. Rev. **135**, 374 [1964].

Let the primary waves be

$$\mathbf{E}_\mathrm{p}(\boldsymbol{r},t) = \sum_\sigma \boldsymbol{E}_\sigma \exp\{-i(\boldsymbol{k}_\sigma\cdot\boldsymbol{r}-\omega_\sigma t)\};\ \boldsymbol{k}_\sigma\cdot\boldsymbol{E}_\sigma=0;$$

$$\boldsymbol{B}_\mathrm{p}(\boldsymbol{r},t) = \sum_\sigma \boldsymbol{B}_\sigma \exp\{-i(\boldsymbol{k}_\sigma\cdot\boldsymbol{r}-\omega_\sigma t)\};\ \boldsymbol{B}_\sigma=\frac{c}{\omega_\sigma}[\boldsymbol{k}_\sigma\times\boldsymbol{E}_\sigma]; \tag{1}$$

$$\sigma=\pm1,\ \pm2;\quad \boldsymbol{E}_{-\sigma}=\boldsymbol{E}_\sigma^*;\quad \boldsymbol{k}_{-\sigma}=-\boldsymbol{k}_{-\sigma};\quad \omega_{-\sigma}=-\omega_\sigma;\quad \boldsymbol{n}_{-\sigma}=\boldsymbol{n}_\sigma;\quad \boldsymbol{k}_\sigma=\boldsymbol{n}_\sigma k_\sigma\,.$$

The plasma consists of N_e electrons and N_i ions with charges q_e, q_i and masses m_e, m_i in the volume V and is described by the functions $F_\alpha(\boldsymbol{r},\boldsymbol{v},t)$, $\alpha=\mathrm{e},\mathrm{i}$:

$$F_\alpha(\boldsymbol{r},\boldsymbol{v},t)=\sum_{i=1}^{N_\alpha}\delta(\boldsymbol{r}-\boldsymbol{r}_i)\,\delta(\boldsymbol{v}-\dot{\boldsymbol{r}}_i) \tag{2}$$

where $\boldsymbol{r}_i$, $\dot{\boldsymbol{r}}_i$ are the position and velocity of the i-th particle. For F_α we have [10, 11]

$$\frac{\partial F_\alpha}{\partial t}+\boldsymbol{v}\cdot\nabla F_\alpha+\frac{q_\alpha}{m_\alpha}\left(\mathsf{E}+\frac{1}{c}[\boldsymbol{v}\times\mathsf{B}]\right)\cdot\nabla_v F_\alpha=0 \tag{3}$$

where $\mathsf{E}(\boldsymbol{r})$, $\mathsf{B}(\boldsymbol{r})$ is the effective self-consistent field at the point $\boldsymbol{r}$. $F_\alpha(\boldsymbol{r},\boldsymbol{v},t)$ depends on the values of the particle cordinates at some initial time, say $t=0$. It is useful to consider quantities averaged over the initial values of

$$\boldsymbol{r}_i(t=0)=\boldsymbol{R}_i;\quad \dot{\boldsymbol{r}}_i(t=0)=\boldsymbol{v}_i\,. \tag{4}$$

With such an averaging procedure we put

$$\langle F_\alpha(\boldsymbol{r},\boldsymbol{v},t)\rangle=f_\alpha(\boldsymbol{r},\boldsymbol{v},t),\quad \langle\mathsf{E}\rangle=\boldsymbol{E},\quad \langle\mathsf{B}\rangle=\boldsymbol{B}\,. \tag{5}$$

The particle structure of the plasma is then contained in the fluctuations

$$\delta f_\alpha(\boldsymbol{r},\boldsymbol{v},t)=F_\alpha-\langle F_\alpha\rangle,\quad \delta\boldsymbol{E}=\mathsf{E}-\langle\mathsf{E}\rangle,\quad \delta\boldsymbol{B}=\mathsf{B}-\langle\mathsf{B}\rangle\,. \tag{6}$$

Averaging (3) there results

$$\frac{\partial f_\alpha}{\partial t}+\boldsymbol{v}\cdot\nabla f_\alpha+\frac{q_\alpha}{m_\alpha}\boldsymbol{A}\cdot\nabla_v f_\alpha=-\frac{q_\alpha}{m_\alpha}\langle\delta\boldsymbol{A}\cdot\nabla\ \delta f_\alpha\rangle\,, \tag{7}$$

$$\frac{\partial\delta f}{\partial t}+\boldsymbol{v}\cdot\nabla\delta f_\alpha+\frac{q_\alpha}{m_\alpha}\boldsymbol{A}\cdot\nabla_v\delta f_\alpha+\frac{q_\alpha}{m_\alpha}\delta\boldsymbol{A}\cdot\nabla_v f_\alpha=-\frac{q_\alpha}{m_\alpha}\{\delta\boldsymbol{A}\cdot\nabla_v\delta f_\alpha-\langle\delta\boldsymbol{A}\cdot\nabla_v\delta f_\alpha\rangle\} \tag{8}$$

with

$$\boldsymbol{A}=\boldsymbol{E}+\frac{1}{c}[\boldsymbol{v}\times\boldsymbol{B}],\quad \delta\boldsymbol{A}=\delta\boldsymbol{E}+\frac{1}{c}[\boldsymbol{v}\times\delta\boldsymbol{B}]\,. \tag{9}$$

(7) is the Vlasov equation with a collisional term. (8) is an equation for the fluctuations. For further simplification the following assumptions are made.

a) Correlations involving more than two particles are neglected, which can be done when there are many particles in a Debye sphere. The rigth hand side of (8) is thus neglected.

b) Coulomb collisions are assumed to be unimportant, and so the right hand side of (7) is set equal to zero.

c) For the plasma without electromagnetic waves we have

$$f_\alpha^{(0)}(\boldsymbol{v})=\frac{N_\alpha}{V}\left(\frac{1}{\pi^{1/2}v_\alpha}\right)^3\exp\{-(\boldsymbol{v}/v_\alpha)^2\}=n_\alpha f_{\alpha0}(\boldsymbol{v}),\qquad v_\alpha^2=\frac{2\varkappa T_\alpha}{m_\alpha}\,. \tag{10}$$

d) The quantities of interest f_α, δf_α, $\boldsymbol{E}$, $\delta\boldsymbol{E}$ etc. may be expanded in powers of the amplitude of the primary waves.

$$f_\alpha(\boldsymbol{r},\boldsymbol{v},t)=f_\alpha^{(0)}+f_\alpha^{(1)}+f_\alpha^{(2)}+\dots\,. \tag{11}$$

[10] Yu. L. Klimontovitch and V. P. Silin, Soviet Phys.-JETP 15, 199 [1962]. — J. M. Dawson and T. Nakayama, Phys. Fluids 9, 252 [1966].
[11] A. Salat, IPP 6/49 [1966].

Nonlinear effects can thus be found in the second order. It is useful to go over to complex FOURIER representation in the following way:

$$f_a(\boldsymbol{r}, \boldsymbol{v}, t) = \frac{1}{(2\pi)^4} \int^* d\omega \int d^3k \exp\{-i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)\} f_a(\omega, \boldsymbol{k}, \boldsymbol{v}),$$
$$f_a(\omega, \boldsymbol{k}, \boldsymbol{v}) = \int_0^\infty dt \int d^3r \exp\{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)\} f_a(\boldsymbol{r}, \boldsymbol{v}, t). \tag{12}$$

The asterisk indicates that the path of integration in the complex ω-plane goes below all singularities of the integrand. With a) and b) there results

$$i(\omega - \boldsymbol{k}\cdot\boldsymbol{v})\, \delta f_a(\omega, \boldsymbol{k}, \boldsymbol{v}) - f_a(t=0, \boldsymbol{k}, \boldsymbol{v}) + \frac{q_a}{m_a} \boldsymbol{A} \circ \nabla_v f_a = 0, \tag{13}$$

$$i(\omega - \boldsymbol{k}\cdot\boldsymbol{v})\, f_a(\omega, \boldsymbol{k}, \boldsymbol{v}) - \delta f_a(t=0, \boldsymbol{k}, \boldsymbol{v}) + \frac{q_a}{m_a} \boldsymbol{A} \circ \nabla_v \delta f_a + \frac{q_a}{m_a} \delta\boldsymbol{A} \circ \nabla_v f_a = 0 \tag{14}$$

where $A \circ B$ stands for the convolution integral $(2\pi)^{-4} \int^* d\omega' \int d^3k'\, A(\omega', k') \cdot B(\omega - \omega', \boldsymbol{k} - \boldsymbol{k}')$. In FOURIER representation MAXWELL's equations can be put in the following form:

$$\boldsymbol{E}(\omega, \boldsymbol{k}) = \frac{E_A(\omega, \boldsymbol{k})}{k^2 c^2 - \omega^2} + \frac{4\pi i}{k^2} \left\{ \boldsymbol{k}\, \varrho(\omega, \boldsymbol{k}) + \frac{\omega}{k^2 c^2 - \omega^2} [\boldsymbol{k} \times [\boldsymbol{k} \times \boldsymbol{j}(\omega, \boldsymbol{k})]] \right\} \tag{15}$$

$$\boldsymbol{E}_A = \frac{-i}{k^2} \{\omega [\boldsymbol{k} \times [\boldsymbol{k} \times \boldsymbol{E}(t=0, \boldsymbol{k})]] + k^2 c [\boldsymbol{k} \times \boldsymbol{B}(t=0, \boldsymbol{k})]\}, \quad k = |\boldsymbol{k}| \tag{16}$$

and

$$\varrho = \sum_\alpha \varrho_a = \sum q_a \int d^3v\, f_a(\boldsymbol{v}), \qquad \boldsymbol{j} = \sum_\alpha \boldsymbol{j}_a = \sum q_a \int d^3v\, \boldsymbol{v}\, f_a(\boldsymbol{v}) \tag{17}$$

and analogously for the fluctuations.

By specifying initial conditions the system (13) – (17) is fully determined.

2. Thermal Plasma

This well-known regime is presented briefly here because the following sections depend closely on the way solutions are found in this section.

After separating into longitudinal and transverse parts the Eqs. (13) – (15) read:

$$\boldsymbol{E}^{(0)}(\omega, \boldsymbol{k}) \equiv 0, \qquad i(\omega - \boldsymbol{k}\cdot\boldsymbol{v})\, \delta f_\alpha^{(0)}(\omega, \boldsymbol{k}, \boldsymbol{v}) + \frac{q_a}{m_a} \delta\boldsymbol{E}^{(0)}(\omega, \boldsymbol{k}) \cdot \nabla_v f_\alpha^{(0)} = \delta f_\alpha^{(0)}(t=0, \boldsymbol{k}, \boldsymbol{v}), \tag{18, 19}$$

$$\boldsymbol{k}\cdot\delta\boldsymbol{E}^{(0)}(\omega, \boldsymbol{k}) = 4\pi\, \varrho^{(0)}(\omega, \boldsymbol{k}), \quad [\boldsymbol{k} \times \delta\boldsymbol{E}^{(0)}(\omega, \boldsymbol{k})] = \frac{[\boldsymbol{k} \times \delta\boldsymbol{E}^{(0)}{}_A]}{k^2 c^2 - \omega^2} - \frac{4\pi i\, \omega}{k^2 c^2 - \omega^2} [\boldsymbol{k} \times \delta\boldsymbol{j}^{(0)}(\omega, \boldsymbol{k})]. \tag{20}$$

Taking moments from (19), (20) there results

$$\boldsymbol{k}\cdot\delta\boldsymbol{E}^{(0)} = 4\pi \sum q_a \int \frac{d^3v}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}} \left\{ \delta f_\alpha^{(0)}(t=0, \boldsymbol{k}, \boldsymbol{v}) - \frac{q_a}{m_a} \delta\boldsymbol{E}^{(0)} \cdot \nabla_v f_\alpha^{(0)} \right\} \tag{21}$$

and from this

$$\boldsymbol{k}\cdot\delta\boldsymbol{E}^{(0)}(\omega, \boldsymbol{k}) = \sum_\alpha \boldsymbol{k}\cdot\delta\boldsymbol{E}_\alpha^{(0)} = \sum_\alpha \frac{4\pi}{\varepsilon(\omega, k)} q_a \int \frac{d^3v}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}} \delta f_\alpha^{(0)}(t=0, \boldsymbol{k}, \boldsymbol{v}) \tag{22}$$

with

$$\varepsilon(\omega, k) = 1 - \sum_\alpha \omega_a^2 \int \frac{d^3v}{(\omega - \boldsymbol{k}\cdot\boldsymbol{v})^2} f_{a0}(\boldsymbol{v}) = 1 - \sum_\alpha \omega_a^2 \int_{-\infty}^{+\infty} \frac{dv}{(\omega - k v)^2} f_{a0}(v), \qquad \omega_a^2 = \frac{4\pi n_a q_a^2}{m_a}. \tag{23}$$

For the approximations to the longitudinal dielectric constant $\varepsilon(\omega, k)$ see Appendix A.

Analogously, from taking the velocity moment it follows that

$$[\boldsymbol{k} \times \delta\boldsymbol{E}^{(0)}(\omega, \boldsymbol{k})] = \frac{[\boldsymbol{k} \times \delta\, \boldsymbol{E}_A^{(0)}]}{\alpha(\omega, k)} + \sum_\alpha [\boldsymbol{k} \times \delta\boldsymbol{E}_\alpha^{(0)}(\omega, \boldsymbol{k})]$$
$$= \frac{[\boldsymbol{k} \times \delta\, \boldsymbol{E}_A^{(0)}]}{\alpha(\omega, k)} + \sum_\alpha \frac{-4\pi}{\alpha(\omega, k)} q_a \int \frac{d^3v\, \omega\, [\boldsymbol{k} \times \boldsymbol{v}]}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}} \delta f_\alpha^{(0)}(t=0, \boldsymbol{k}, \boldsymbol{v}) \tag{24}$$

with
$$\alpha(\omega, k) = k^2 c^2 - \omega^2 + \sum_\alpha \omega_\alpha{}^2 \int_{-\infty}^{+\infty} \frac{\mathrm{d}v\, \omega}{\omega - k v} f_{\alpha 0}(v) . \tag{25}$$

Again, taking moments of (19), the following density and velocity fluctuations occur:
$$\delta N_\alpha^{(0)}(\omega, \boldsymbol{k}) = \int \mathrm{d}^3 v\, \delta f_\alpha^{(0)} = \frac{1}{4\pi i q_\alpha} \sum_\beta \boldsymbol{k}\cdot\delta\boldsymbol{E}_\beta^{(0)}(\omega, \boldsymbol{k})\, a_{\alpha\beta}(\omega, k), \tag{26}$$

$$\boldsymbol{k}\cdot\delta\boldsymbol{V}_\alpha^{(0)}(\omega, \boldsymbol{k}) = \frac{\omega}{4\pi i q_\alpha} \sum_\beta \boldsymbol{k}\cdot\delta\boldsymbol{E}_\beta^{(0)}\, a_{\alpha\beta} + i \int \mathrm{d}^3 v\, \delta f_\alpha^{(0)}(t=0, \boldsymbol{k}, \boldsymbol{v}), \tag{27}$$

$$[\boldsymbol{k}\times\delta\boldsymbol{V}_\alpha^{(0)}(\omega, \boldsymbol{k})] = -\frac{1}{4\pi i \omega q_\alpha} \sum_\beta [\boldsymbol{k}\times\delta\boldsymbol{E}_\beta^{(0)}]\, b_{\alpha\beta}(\omega k) - \frac{i q_\alpha [\boldsymbol{k}\times\delta\boldsymbol{E}_A^{(0)}]}{m_\alpha\, \alpha(\omega, k)} \int \frac{\mathrm{d}v\, f_\alpha^{(0)}(v)}{\omega - k v} \tag{28}$$

with
$$a_{\alpha\beta}(\omega, k) = \varepsilon(\omega, k)\, \delta_{\alpha\beta} + \omega_\alpha{}^2 \int \frac{\mathrm{d}v\, f_{\alpha 0}(v)}{(\omega - k v)^2}, \qquad b_{\alpha\beta}(\omega, k) = \alpha(\omega, k)\, \delta_{\alpha\beta} - \omega_\alpha{}^2 \int \frac{\mathrm{d}v\, \omega\, f_{\alpha 0}(v)}{\omega - k v} . \tag{29}$$

The well-known formula for thermal density fluctuations follows from (26), (22) by averaging over the initial values (4) of $\boldsymbol{r}_i$, $\dot{\boldsymbol{r}}_i$. Using
$$\lim_{\gamma\to 0} \frac{\gamma}{x^2+\gamma^2} = \pi\, \delta(x), \qquad \gamma = \operatorname{Im}\omega \tag{30}$$
the result for $\boldsymbol{k} \neq 0$ is
$$\frac{1}{V} \lim_{\gamma\to 0} \gamma \langle |\delta N_\alpha^{(0)}(\omega, \boldsymbol{k})|^2 \rangle = \frac{1}{V} \lim_{\gamma\to 0} \gamma \left\langle \left| \sum_\beta \sum_{i=1}^{N\beta} \frac{\exp\{i \boldsymbol{k}\cdot\boldsymbol{R}_i\}}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}_i} \frac{a_{\alpha\beta}(\omega, k)}{\varepsilon(\omega, k)} \right|^2 \right\rangle = \sum_\beta \frac{\pi n_\beta}{k} f_{\beta 0}\left(\frac{\omega}{k}\right) \left| \frac{a_{\alpha\beta}(\omega, k)}{\varepsilon(\omega, k)} \right|^2 \tag{31}$$

In the following, the transverse part of thermal fluctuations is neglected.

3. Linear Processes

From the linearized VLASOV Eq. (13) the frequencies ω_σ and wave numbers k_σ of waves (1) are coupled by the dispersion relation
$$\alpha(\omega_\sigma, k_\sigma) = k_\sigma{}^2 c^2 - \omega_\sigma{}^2 + \sum \omega_\alpha{}^2 = 0 . \tag{32}$$

The plasma together with the waves oscillates according to
$$\begin{aligned} \boldsymbol{E}^{(1)}(\omega, \boldsymbol{k}) &= \sum_\sigma \boldsymbol{E}_\sigma \frac{-i(2\pi)^3 \delta(\boldsymbol{k} - \boldsymbol{k}_\sigma)}{\omega - \omega_\sigma}, \\ f_\alpha^{(1)}(\omega, \boldsymbol{k}, \boldsymbol{v}) &= \sum_\sigma f_{\alpha\sigma} \frac{-i(2\pi)^3 \delta(\boldsymbol{k} - \boldsymbol{k}_\sigma)}{\omega - \omega_\sigma}, \qquad f_{\alpha\sigma} = \frac{i q_\alpha}{m_\alpha \omega_\sigma} \boldsymbol{E}_\sigma \cdot \nabla_v f_\alpha^{(0)}(\boldsymbol{v}), \\ N_\alpha^{(1)}(\omega, \boldsymbol{k}) &= \int \mathrm{d}^3 v\, f_\alpha^{(1)}(\omega, \boldsymbol{k}, \boldsymbol{v}) \equiv 0 \end{aligned} \tag{33}$$

results which follow trivially from (13), (15), (16) and the assumption $|\omega/\boldsymbol{k}\cdot\boldsymbol{v}| \gg 1$, which for $\omega = \omega_\sigma$, $\boldsymbol{k} = \boldsymbol{k}_\sigma$ and (32) is automatically fulfilled.

Besides these collective effects, the primary waves produce fluctuations in the plasma.

The linear approximation to Eq. (14) reads
$$i(\omega - \boldsymbol{k}\cdot\boldsymbol{v})\, \delta f_\alpha^{(1)}(\omega, \boldsymbol{k}, \boldsymbol{v}) + \frac{q_\alpha}{m_\alpha} \delta\boldsymbol{E}^{(1)}\cdot\nabla_v f_\alpha^{(0)} = \delta f_\alpha^{(1)}(t=0) - \frac{q_\alpha}{m_\alpha} \boldsymbol{A}^{(1)} \circ \nabla_v \delta f_\alpha^{(0)} - \frac{q_\alpha}{m_\alpha} \delta\boldsymbol{E}^{(0)} \circ \nabla_v f_\alpha^{(1)} . \tag{34}$$

In comparison to the thermal case, only the inhomogeneous right hand side has changed. Since also MAXWELL's equations are the same in any order, the solution to Eq. (34) can immediately be written down by comparing with the last section. Choosing $\delta f_\alpha^{(1)}(t=0) = \delta\boldsymbol{E}^{(1)}(t=0) = 0$, omitting the $\boldsymbol{B}$-field of the waves and unfolding the convolution integral, there results with the help of (22) and (24)
$$\boldsymbol{k}\cdot\delta\boldsymbol{E}^{(1)}(\omega, \boldsymbol{k}) = \frac{4\pi}{\varepsilon(\omega, k)} \sum_\alpha \sum_\sigma \frac{q_\alpha{}^2}{m_\alpha} \int \frac{\mathrm{d}^3 v}{(\omega - \boldsymbol{k}\cdot\boldsymbol{v})^2} \boldsymbol{k}\cdot\{\boldsymbol{E}_\sigma\, \delta f_\alpha^{(0)}(\omega - \omega_\sigma, \boldsymbol{k} - \boldsymbol{k}_\sigma, \boldsymbol{v}) + f_{\alpha\sigma}\, \delta\boldsymbol{E}^{(0)}(\omega - \omega_\sigma, \boldsymbol{k} - \boldsymbol{k}_\sigma)\}, \tag{35}$$

$$[\boldsymbol{k}\times\delta\boldsymbol{E}^{(1)}(\omega,\boldsymbol{k})] = -\frac{4\pi}{\alpha(\omega,k)}\sum_\alpha\sum_\sigma\frac{q_\alpha^2}{m_\alpha}\int\frac{d^3v\,\omega[\boldsymbol{k}\times\boldsymbol{v}]}{(\omega-\boldsymbol{k}\cdot\boldsymbol{v})^2}\,\boldsymbol{k}\cdot\{\boldsymbol{E}_\sigma\,\delta f_\alpha^{(0)}(\omega-\omega_\sigma,\boldsymbol{k}-\boldsymbol{k}_\sigma)$$
$$+f_{\alpha\sigma}\,\delta\boldsymbol{E}^{(0)}(\omega-\omega_\sigma,\boldsymbol{k}-\boldsymbol{k}_\sigma)\} - \frac{4\pi}{\alpha(\omega,k)}\sum_\alpha\sum_\sigma\frac{q_\alpha^2}{m_\alpha}\int\frac{d^3v\,\omega}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}} \tag{36}$$
$$\cdot\,[\boldsymbol{k}\times\{\boldsymbol{E}_\sigma\,\delta f_\alpha^{(0)}(\omega-\omega_\sigma,\boldsymbol{k}-\boldsymbol{k}_\sigma)+f_{\alpha\sigma}\,\delta\boldsymbol{E}^{(0)}(\omega-\omega_\sigma,\boldsymbol{k}-\boldsymbol{k}_\sigma)\}]\,.$$

(35), (36) considerably simplify under the restriction $|\omega/\boldsymbol{k}\cdot\boldsymbol{v}|\gg 1$. Then the contribution of the pole $\boldsymbol{k}\cdot\boldsymbol{v}=\omega$ is exponentially small, and from the expansion of the denominator there results

$$\boldsymbol{k}\cdot\delta\boldsymbol{E}^{(1)}(\omega,\boldsymbol{k}) = \sum_\alpha\boldsymbol{k}\cdot\delta\boldsymbol{E}_\alpha^{(1)} = \frac{4\pi}{\varepsilon(\omega,k)}\sum_\alpha\sum_\sigma\frac{q_\alpha^2}{m_\alpha}\frac{1}{\omega^2}\,\boldsymbol{k}\cdot\boldsymbol{E}_\sigma$$
$$\cdot\left\{\delta N_\alpha^{(0)}(\omega-\omega_\sigma,\boldsymbol{k}-\boldsymbol{k}_\sigma)+\sum_\beta\frac{\omega\,\omega_\sigma}{2\,\omega_\alpha^2}\frac{q_\beta}{q_\alpha}\,\boldsymbol{k}\cdot\boldsymbol{l}_\sigma\,l_\sigma^{-2}\,\delta N_\beta^{(0)}(\omega-\omega_\sigma,\boldsymbol{k}-\boldsymbol{k}_\sigma)\right\} \tag{37}$$

$$[\boldsymbol{k}\times\delta\boldsymbol{E}^{(1)}(\omega,\boldsymbol{k})] = -\frac{4\pi}{\alpha(\omega,k)}\sum_\alpha\sum_\sigma\frac{q_\alpha^2}{m_\alpha}\Big\{[\boldsymbol{k}\times\boldsymbol{E}_\sigma]\,\delta N_\alpha^{(0)}(\omega-\omega_\sigma,\boldsymbol{k}-\boldsymbol{k}_\sigma)+$$
$$+\sum_\beta\frac{\omega_\alpha^2}{\omega\,\omega_\sigma}\frac{q_\beta}{q_\alpha}\,l_\sigma^{-2}([\boldsymbol{k}\times\boldsymbol{E}_\sigma]\,\boldsymbol{k}\cdot\boldsymbol{l}_\sigma+\boldsymbol{k}\cdot\boldsymbol{E}_\sigma[\boldsymbol{k}\times\boldsymbol{l}_\sigma])\,\delta N_\beta^{(0)}(\omega-\omega_\sigma,\boldsymbol{k}-\boldsymbol{k}_\sigma) \tag{38}$$

with
$$\boldsymbol{l}_\sigma=\boldsymbol{k}-\boldsymbol{k}_\sigma\,,\qquad l_\sigma=|\boldsymbol{l}_\sigma|\,. \tag{39}$$

(38) is the well-known formula for the scattering of electromagnetic waves [12] on density fluctuations in a plasma, written in $\boldsymbol{k}$-space. The second term describes scattering of thermal fields on induced plasma polarisation. Since the poles of $\delta N_\alpha^{(0)}(\omega-\omega_\sigma)$ are close to $\omega=\omega_\sigma$, this term is smaller than the first by a factor of $\omega_\alpha^2/\omega_\sigma^2$ and is generally omitted for $\omega_\sigma^2\gg\omega_\alpha^2$.

The connection between induced density etc. fluctuations and fluctuations of the electrical field because of the analogous form of the equations is the same as in the thermal case:

$$\delta N_\alpha^{(1)}(\omega,\boldsymbol{k}) = \frac{1}{4\pi i\,q_\alpha}\sum_\beta\boldsymbol{k}\cdot\delta\boldsymbol{E}_\beta^{(1)}(\omega,\boldsymbol{k})\,a_{\alpha\beta}(\omega,k) \tag{40}$$

etc. as in (27), (28).

4. Nonlinear Processes

Nonlinear collective processes, e. g. periodical density changes induced by two electro-magnetic waves were discussed in [5, 6, 8]. There was proposed [6] to detect these density variations by aiming a third beam at them and looking for the light which is „scattered" into very special angles only. It is therefore sufficient here to summarize very briefly the collective processes. Eq. (13) reads for $f_\alpha^{(2)}(t=0)=0$

$$i(\omega-\boldsymbol{k}\cdot\boldsymbol{v})\,f_\alpha^{(2)}(\omega,\boldsymbol{k},\boldsymbol{v})+\frac{q_\alpha}{m_\alpha}\boldsymbol{E}^{(2)}\cdot\nabla_v f_\alpha^{(0)} = -\frac{q_\alpha}{m_\alpha}\boldsymbol{A}^{(1)}\circ\nabla_v f_\alpha^{(1)}. \tag{41}$$

Again by analogy with section 2, the solution is

$$\boldsymbol{k}\cdot\boldsymbol{E}^{(2)}(\omega,\boldsymbol{k}) = -\frac{4\pi}{\varepsilon(\omega,k)}\sum_\alpha\frac{q_\alpha^2}{m_\alpha}\int\frac{d^3v}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}}\left\{\boldsymbol{E}^{(1)}+\frac{1}{c}[\boldsymbol{v}\times\boldsymbol{B}^{(1)}]\right\}\circ\nabla_v f_\alpha^{(1)}. \tag{42}$$

For $|\omega/\boldsymbol{k}\cdot\boldsymbol{v}|\gg 1$ i. e. $|\omega_s|/k_s v_\alpha\gg 1$ (43) the expansion gives

$$\boldsymbol{E}^{(2)}(\omega,\boldsymbol{k}) = \sum_\alpha\sum_{\sigma,\tau}\boldsymbol{E}_{s,\alpha}\frac{-i(2\pi)^3}{\omega-\omega_s}\,\delta(\boldsymbol{k}-\boldsymbol{k}_s),\qquad \boldsymbol{E}_{s,\alpha} = -\frac{i\,q_\alpha}{m_\alpha}\frac{\omega_\alpha^2}{2\,\omega_\sigma\,\omega_\tau}\frac{1}{\omega_s^2}\frac{1}{\varepsilon(\omega_s,k_s)}\,\boldsymbol{k}_s\,\boldsymbol{E}_\sigma\cdot\boldsymbol{E}_\tau \tag{44}$$

with
$$\omega_s=\omega_\sigma+\omega_\tau\,,\qquad \boldsymbol{k}_s=\boldsymbol{k}_\sigma+\boldsymbol{k}_\tau\,,\qquad k_s=|\boldsymbol{k}_s|\,. \tag{45}$$

The transverse part of $\boldsymbol{E}^{(2)}$ vanishes identically. The amplitude of $\boldsymbol{E}^{(2)}$ may become large, when the difference frequency of ω_s is close to the plasma frequency $\omega_p=(\Sigma\,\omega_\alpha^2)^{1/2}$.

[12] E. E. SALPETER, Phys. Rev. 120. 1528 [1960].

For the nonlinear fluctuations (14) gives

$$i(\omega-\boldsymbol{k}\cdot\boldsymbol{v})\,\delta f_\alpha^{(2)}(\omega,\boldsymbol{k},\boldsymbol{v})+\frac{q_\alpha}{m_\alpha}\,\delta\boldsymbol{E}^{(2)}\cdot\nabla_v f_\alpha^{(0)}$$
$$=\delta f_\alpha^{(2)}(t=0,\boldsymbol{k},\boldsymbol{v})-\frac{q_\alpha}{m_\alpha}\{\boldsymbol{A}^{(1)}\circ\nabla_v\delta f_\alpha^{(1)}+\delta\boldsymbol{A}^{(1)}\circ\nabla_v f_\alpha^{(1)}+\boldsymbol{E}^{(2)}\circ\nabla_v\delta f_\alpha^{(0)}+\delta\boldsymbol{E}^{(0)}\circ\nabla_v f_\alpha^{(2)}\}\,. \tag{46}$$

For $\delta f_\alpha^{(2)}(t=0)=\delta\boldsymbol{E}^{(2)}(t=0)=0$ and with MAXWELL's equations the transverse field follows from analogy with the thermal regime (24):

$$[\boldsymbol{k}\times\delta\boldsymbol{E}^{(2)}(\omega,\boldsymbol{k})]=\frac{4\pi}{\alpha(\omega,k)}\sum_\alpha\frac{q_\alpha^2}{m_\alpha}\int\frac{d^3v\,\omega[\boldsymbol{k}\times\boldsymbol{v}]}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}}\cdot\{\boldsymbol{A}^{(1)}\circ\nabla_v\delta f_\alpha^{(1)}$$
$$+\delta\boldsymbol{A}^{(1)}\circ\nabla_v f_\alpha^{(1)}+\boldsymbol{E}^{(2)}\circ\nabla_v\delta f_\alpha^{(0)}+\delta\boldsymbol{E}^{(0)}\circ\nabla_v f_\alpha^{(2)}\}\,. \tag{47}$$

As can be seen by the pole $\alpha(\omega,k)=0$ (47) contains fluctuating electric (electromagnetic) fields that propagate as waves in the plasma, and that are radiated away when the plasma volume is finite. This nonlinearly scattered light comes from four different processes: a) Primary waves $\boldsymbol{A}^{(1)}$ are scattered by induced fluctuations $\delta f_\alpha^{(1)}$. b) Induced fluctuations of the electric field $\delta\boldsymbol{A}^{(1)}$ are scattered by the plasma polarisation $f_\alpha^{(1)}$. c) Nonlinear fields $\boldsymbol{E}^{(2)}$ are scattered by thermal fluctuations $\delta f_\alpha^{(0)}$. d) Thermal fields are scattered by the polarization $f_\alpha^{(2)}$. The scattered light, (47), due to the stochastic fluctuation, has a finite time and length of coherence. It may be called partially coherent or incoherent.

Since $\alpha(\omega,k)=0$ for real ω only permits $|\omega/k|\geqq c$, the denominator of (47) can be expanded to give

$$\begin{aligned}[\boldsymbol{k}\times\delta\boldsymbol{E}^{(2)}(\omega,\boldsymbol{k})]=&-\frac{4\pi}{\alpha(\omega,k)}\sum_\alpha\sum_{\sigma,\tau}\frac{q_\alpha^2}{m_\alpha}\Big\{[\boldsymbol{k}\times\boldsymbol{E}_\sigma]\,\delta N_\alpha^{(1)}(\omega-\omega_\sigma)+\frac{1}{\omega}\boldsymbol{k}\cdot\boldsymbol{E}_\sigma[\boldsymbol{k}\times\delta\boldsymbol{V}_\alpha^{(1)}(\omega-\omega_\sigma)]\\&+\frac{1}{\omega}[\boldsymbol{k}\times\boldsymbol{E}_\sigma]\,\boldsymbol{k}\cdot\delta\boldsymbol{V}_\alpha^{(1)}(\omega-\omega_\sigma)+\frac{1}{\omega}\boldsymbol{k}\cdot\delta\boldsymbol{E}^{(1)}(\omega-\omega_\sigma)[\boldsymbol{k}\times\boldsymbol{V}_{\sigma,\alpha}]\\&+\frac{1}{\omega}[\boldsymbol{k}\times\delta\boldsymbol{E}^{(1)}(\omega-\omega_\sigma)]\,\boldsymbol{k}\cdot\boldsymbol{V}_{\sigma,\alpha}+\frac{1}{c}[\boldsymbol{k}\times[\delta\boldsymbol{V}_\alpha^{(1)}(\omega-\omega_\sigma)\times\boldsymbol{B}_\sigma]]\\&+\frac{1}{c}[\boldsymbol{k}\times[\boldsymbol{V}_{\sigma,\alpha}\times\delta\boldsymbol{B}^{(1)}(\omega-\omega_\sigma)]]+[\boldsymbol{k}\times\boldsymbol{E}_s]\,\delta N_\alpha^{(0)}(\omega-\omega_s)\\&+[\boldsymbol{k}\times\delta\boldsymbol{E}^{(0)}(\omega-\omega_s)]\,N_{s,\alpha}+\frac{1}{\omega}\boldsymbol{k}\cdot\delta\boldsymbol{E}^{(0)}(\omega-\omega_s)[\boldsymbol{k}\times\boldsymbol{V}_{s,\alpha}]\\&+\frac{1}{\omega}[\boldsymbol{k}\times\delta\boldsymbol{E}^{(0)}(\omega-\omega_s)]\,\boldsymbol{k}\cdot\boldsymbol{V}_{s,\alpha}\Big\}.\end{aligned} \tag{48}$$

The frequency spectrum of (48) is centered around the frequencies $\omega_s=\omega_\sigma+\omega_\tau$. Using the results of sections 2 and 3 gives two types of terms:

$$[\boldsymbol{k}\times\delta\boldsymbol{E}^{(2)}(\omega,\boldsymbol{k})]=[\boldsymbol{k}\times\delta\boldsymbol{E}_{\mathrm{I}}]+[\boldsymbol{k}\times\delta\boldsymbol{E}_{\mathrm{II}}] \tag{49}$$

with

$$\begin{aligned}[\boldsymbol{k}\times\delta\boldsymbol{E}_{\mathrm{I}}]=&\frac{4\pi i}{\alpha(\omega,k)}\sum_{\sigma,\tau}\sum_{\alpha,\beta,\gamma}\frac{q_\beta^2}{m_\beta}\frac{q_\alpha}{m_\alpha}\frac{1}{\varepsilon(\omega-\omega_\sigma)}\frac{1}{\omega_\tau^2}\frac{1}{l_\sigma^2}\,\boldsymbol{l}_\sigma\cdot\boldsymbol{E}_\tau\\&\cdot\Big\{l_\sigma^2[\boldsymbol{k}\times\boldsymbol{E}_\sigma]+\frac{\omega_\tau}{\omega_s}[\boldsymbol{k}\times\boldsymbol{l}_\sigma]\,\boldsymbol{k}\cdot\boldsymbol{E}_\sigma+\frac{\omega_\tau}{\omega_s}\boldsymbol{k}\cdot\boldsymbol{l}_\sigma[\boldsymbol{k}\times\boldsymbol{E}_\sigma]+\frac{\omega_\tau}{c}[\boldsymbol{k}\times[\boldsymbol{l}_\sigma\times\boldsymbol{B}_\sigma]]\cdot a_{\alpha\beta}(\omega-\omega_s)\\&+\frac{\omega_\alpha^2}{\omega_\sigma\,\omega_s}([\boldsymbol{k}\times\boldsymbol{l}_\sigma]\,\boldsymbol{k}\cdot\boldsymbol{E}_\sigma+\boldsymbol{k}\cdot\boldsymbol{l}_\sigma[\boldsymbol{k}\times\boldsymbol{E}_\sigma])\Big\}\\&\cdot\Big\{\delta_{\beta\gamma}+2\frac{\omega_\beta^2}{\omega_\tau^2}\frac{q_\gamma}{q_\beta}\frac{1}{l_s^2}\,\boldsymbol{l}_\sigma\cdot\boldsymbol{l}_s\Big\}\,\delta N_\gamma^{(0)}(\omega-\omega_s,\boldsymbol{k}-\boldsymbol{k}_s),\end{aligned} \tag{50}$$

$$\begin{aligned}[\boldsymbol{k}\times\delta\boldsymbol{E}_{\mathrm{II}}]=&\frac{4\pi i}{\alpha(\omega,k)}\sum_{\sigma,\tau}\sum_{\alpha,\beta,\gamma}\frac{q_\alpha^2}{m_\alpha}\frac{q_\beta}{m_\beta}\frac{1}{\varepsilon(\omega_s,k_s)}\frac{\omega_\beta^2}{\omega_s^2}\frac{1}{2\,\omega_\sigma\,\omega_\tau}\,\boldsymbol{E}_\sigma\cdot\boldsymbol{E}_\tau\\&\cdot\Big\{\frac{q_\gamma}{q_\alpha}\frac{1}{l_s^2}(k_s^2[\boldsymbol{k}\times\boldsymbol{l}_s]+\boldsymbol{k}\cdot\boldsymbol{l}_s[\boldsymbol{k}\times\boldsymbol{k}_s]+\boldsymbol{k}\cdot\boldsymbol{k}_s[\boldsymbol{k}\times\boldsymbol{l}_s])\,a_{\alpha\beta}(\omega_s,k_s)\\&+\delta_{\alpha\gamma}[\boldsymbol{k}\times\boldsymbol{k}_s]\Big\}\,\delta N_\gamma^{(0)}(\omega-\omega_s,\boldsymbol{k}-\boldsymbol{k}_s).\end{aligned} \tag{51}$$

The origin of $[\boldsymbol{k}\times\delta\boldsymbol{E}_{\mathrm{I}}]$ are processes of types a) and b). $[\boldsymbol{k}\times\delta\boldsymbol{E}_{\mathrm{II}}]$ comes from processes c) and d).

The fluctuations $\delta \boldsymbol{E}^{(1)}$, $\delta \boldsymbol{B}^{(1)}$, $\delta \boldsymbol{V}_\alpha^{(1)}$ have been replaced by their longitudinal parts. In [11] the contribution of the transverse parts is shown to be negligible, for $\omega_\sigma^2 \gg \omega_\alpha^2$. Depending on the value of the frequency ω_s, two different approximations to the scattered field are discussed in the next sections.

5. Nonresonant Case

If ω_s is not near the plasma frequency ω_p and if, moreover,

$$\omega_s^2 \gg \omega_\alpha^2 \tag{52}$$

only parts of $[\boldsymbol{k} \times \delta \boldsymbol{E}_\mathrm{I}]$ contribute effectively to the scattering:

$$\begin{aligned}[\boldsymbol{k} \times \delta \boldsymbol{E}^{(2)}(\omega, \boldsymbol{k})] = \frac{4 \pi i}{\varepsilon(\omega, k)} \sum_{\sigma,\tau} \sum_{\alpha} \frac{q_\alpha^3}{m_\alpha^2} \frac{1}{\omega_\tau \, \omega_s} \Big\{ [\boldsymbol{k} \times \boldsymbol{E}_\sigma] \, \boldsymbol{l}_\sigma \cdot \boldsymbol{E}_\tau \frac{\omega_s}{\omega_\tau} + \boldsymbol{k} \cdot \boldsymbol{E}_\sigma [\boldsymbol{k} \times \boldsymbol{E}_\tau] \\ + [\boldsymbol{k} \times \boldsymbol{E}_\sigma] \, \boldsymbol{k} \cdot \boldsymbol{E}_\tau + [\boldsymbol{k} \times [\boldsymbol{E}_\tau \times [\boldsymbol{k}_\sigma \times \boldsymbol{E}_\sigma]]] \frac{\omega_s}{\omega_\sigma} \Big\} \, \delta N_\alpha^{(0)} (\omega - \omega_s, \boldsymbol{k} - \boldsymbol{k}_\sigma) .\end{aligned} \tag{53}$$

Eq. (53) describes the scattering of incident waves at induced density and velocity fluctuations. The energy, emitted from the plasma per unit of time into the solid angle $\mathrm{d}\Omega$ and the frequency interval $\mathrm{d}\omega$

$$\mathrm{d}I(\omega, \boldsymbol{n}) = \frac{c}{4 \pi} r^2 \lim_{\gamma \to 0} \frac{\gamma}{\pi} \left| [\delta \boldsymbol{E}^{(2)}(\omega, \boldsymbol{r}) \times \delta \boldsymbol{B}^{(2)}(\omega, \boldsymbol{r})] \right| \mathrm{d}\omega \, \mathrm{d}\Omega \tag{54}$$

is thus found on averaging over the initial conditions of the plasma (see Appendix B) to be:

$$\begin{aligned}\langle \mathrm{d}I(\omega, \boldsymbol{n}) \rangle = \left(\frac{e^2}{m_\mathrm{e} c^2}\right)^2 \frac{c}{4 \pi} \sum_{\sigma,\tau} \Big| \Big[\boldsymbol{n} \times \Big[\boldsymbol{n} \times \Big\{ \boldsymbol{E}_\sigma \frac{1}{\omega_\tau} (\boldsymbol{n} \, \omega_s - \boldsymbol{n}_\sigma \, \omega_\sigma) \cdot \boldsymbol{E}_\tau \\ + \boldsymbol{E}_\tau \, \boldsymbol{n} \cdot \boldsymbol{E}_\sigma + \boldsymbol{E}_\sigma \, \boldsymbol{n} \cdot \boldsymbol{E}_\tau + [\boldsymbol{E}_\tau \times [\boldsymbol{n}_\sigma \times \boldsymbol{E}_\sigma]] \Big\} \Big] \Big] \Big|^2 \\ \cdot \left(\frac{e}{m_\mathrm{e} \, \omega_\tau \, c}\right)^2 (2 - \delta_{\sigma\tau}) \cdot \lim_{\gamma \to 0} \frac{\gamma}{\pi} \langle | \delta N_\mathrm{e}^{(0)} (\omega - \omega_s, \tilde{\boldsymbol{k}} - \boldsymbol{k}_s) |^2 \rangle \, \mathrm{d}\omega \, \mathrm{d}\Omega\end{aligned} \tag{55}$$

with

$$\tilde{\boldsymbol{k}} = \tilde{k} \, \boldsymbol{n} , \qquad \boldsymbol{n} = \boldsymbol{r}/r , \qquad \tilde{k} \, c = (\omega^2 - \omega_\mathrm{p}^2)^{1/2} \operatorname{sign} \operatorname{Re} \omega . \tag{56}$$

The spectrum of the scattered light is given by the electron density fluctuations, as in the linear case. The angular distribution differs from a dipole characteristic. The order of magnitude of the intensity is smaller than the linear one by a factor η:

$$\eta = (e \, E_\sigma / m_\mathrm{e} \, \omega_\sigma \, c)^2 . \tag{57}$$

6. Resonant Case

Let the frequencies ω_1, ω_2 be such that the difference frequency is close to the plasma frequency:

$$| \omega_s | = | \omega_1 - \omega_2 | \approx \omega_\mathrm{p} \tag{58}$$

The amplitudes of the collective oscillations $\boldsymbol{E}^{(2)}$, $N_\alpha^{(2)}$, $\boldsymbol{V}_\alpha^{(2)}$ will then be high, and so parts of $[\boldsymbol{k} \times \delta \boldsymbol{E}_\mathrm{II}]$ dominate in the scattering process. The low frequency wing of the scattered light will not be observable, however, whenever it drops below the plasma frequency. For $m_\mathrm{e} \ll m_\mathrm{i}$ Eq. (51) becomes

$$\begin{aligned}[\boldsymbol{k} \times \delta \boldsymbol{E}^{(2)}(\omega, \boldsymbol{k})] = \frac{4 \pi i}{\varepsilon(\omega, k)} \sum_{\sigma,\tau} \frac{q_\mathrm{e}^3}{m_\mathrm{e}^2} \frac{\boldsymbol{E}_\sigma \cdot \boldsymbol{E}_\tau}{2 \, \omega_\sigma \, \omega_\tau} [\boldsymbol{k} \times \boldsymbol{k}_s] \times \\ \times \left\{ \delta N_\mathrm{e}^{(0)} (\omega - \omega_s) + \left(1 - \frac{2 \, k_s^2}{l_s^2}\right) \frac{1}{q_\mathrm{e}} \left(q_\mathrm{e} \, \delta N_\mathrm{e}^{(0)} (\omega - \omega_s) + q_\mathrm{i} \, \delta N_\mathrm{i}^{(0)} (\omega - \omega_s)\right) \right\} .\end{aligned} \tag{59}$$

Because of

$$k^2 c^2 \to \tilde{k}^2 c^2 \lesssim \omega_\mathrm{p}^2 \quad \text{and} \quad k_s^2 \, c^2 \approx (\boldsymbol{n}_1 \, \omega_1 - \boldsymbol{n}_2 \, \omega_2)^2$$

it follows that

$$\frac{k_s^2}{k^2} \gtrsim 1 + \frac{\omega_1 \, \omega_2}{\omega_\mathrm{p}^2} 4 \sin^2 \frac{\alpha_{12}}{2} \gg 1 \tag{60}$$

except for nearly parallel primary beams. α_{12} is the angle between beam 1 and beam 2. With (60) the electron density fluctuations disappear from (59):

$$[\boldsymbol{k} \times \delta \boldsymbol{E}^{(2)}] = -\frac{4\pi i}{\alpha(\omega, k)} \sum_{\substack{\sigma,\tau = \\ \pm(1-2)}} \frac{q_e^3}{m_e^2} \frac{\boldsymbol{E}_\sigma \cdot \boldsymbol{E}_\tau}{2\,\omega_\sigma\,\omega_\tau} [\boldsymbol{k} \times \boldsymbol{k}_s] \frac{-q_i}{q_e} \frac{1}{\varepsilon(\omega_s, k_s)} \delta N_i^{(0)}(\omega - \omega_s, \boldsymbol{k} - \boldsymbol{k}_s). \tag{61}$$

The POYNTING vector gives the energy emitted into the solid angle $d\Omega$ and the frequency interval $d\omega$ (see Appendix B)

$$\langle dI(\omega, \boldsymbol{n}) \rangle = \left(\frac{e^2}{m_e c^2}\right)^2 \frac{c}{4\pi} \sum_{\sigma,\tau} |\boldsymbol{E}_\sigma \cdot \boldsymbol{E}_\tau|^2 \left(\frac{e}{m_e\,\omega_\sigma\, c}\right)^2 \left|\frac{\omega_\sigma}{\omega_\tau}\right| \left(\frac{q_i}{q_e}\right)^2 \sin^2(\boldsymbol{n}, \boldsymbol{n}_s)$$
$$\cdot 2 \sin^2 \frac{\alpha_{12}}{2} \frac{1}{|\varepsilon(\omega_s, k_s)|^2} \left(\frac{\omega^2 - \omega_p^2}{\omega^2}\right)^{\frac{1}{2}} \lim_{\gamma \to 0} \frac{\gamma}{\pi} \langle |\delta N_i^{(0)}(\omega - \omega_s, \tilde{\boldsymbol{k}} - \boldsymbol{k}_s)|^2 \rangle \, d\omega \, d\Omega. \tag{62}$$

The order of magnitude of the scattering cross section is higher than in the nonresonant case by a factor of $(1 - \omega_p^2/\omega^2)^{1/2} \cdot |\varepsilon(\omega_s, k_s)|^{-2}$ and "higher" than the linear cross section by $(1 - \omega_p^2/\omega^2)^{1/2} \cdot |\varepsilon(\omega_s, k_s)|^{-2} \cdot \eta$. The actual amplitude of the resonance in $|\varepsilon|^{-2}$ depends on the LANDAU damping or collisional damping, whichever is the larger. The frequency spectrum of (62) is given by the ion density fluctuations. From (31):

$$\lim_{\gamma \to 0} \frac{\gamma}{\pi} \langle |\delta N_i^{(0)}(\Delta\omega, \Delta\boldsymbol{k})|^2 \rangle = \frac{V}{\Delta k} \left(n_i f_{i0}\left(\frac{\Delta\omega}{\Delta k}\right) \left| \frac{1 - \omega_e^2 \int \frac{dv\, f_{e0}(v)}{(\Delta\omega - \Delta k\, v)^2}}{1 - \sum \omega_\alpha^2 \int \frac{dv\, f_{\alpha 0}(v)}{(\Delta\omega - \Delta k\, v)^2}} \right|^2 \right.$$
$$\left. + n_e f_{e0}\left(\frac{\Delta\omega}{\Delta k}\right) \left| \frac{\omega_i^2 \int \frac{dv\, f_{i0}(v)}{(\Delta\omega - \Delta k\, v)^2}}{1 - \sum \omega_\alpha^2 \int \frac{dv\, f_{\alpha 0}(v)}{(\Delta\omega - \Delta k\, v)^2}} \right|^2 \right) =: S(\Delta\omega, \boldsymbol{n}) \tag{63}$$

with $$\Delta\omega = \omega - \omega_s, \quad \Delta k = |\Delta\boldsymbol{k}| = |\boldsymbol{k} - \boldsymbol{k}_s|. \tag{64}$$

It seems appropriate at this stage to discuss briefly the frequency dependence of the ion fluctuations. On the assumption that T_e and T_i are not too extremely different the following approximations are found

a) $|\Delta\omega|/\Delta k\, v_a \ll 1, \quad \alpha = e, i$

Using Eq. (A 4)

$$S(\Delta\omega, \boldsymbol{n}) = \frac{V}{\Delta k} \left\{ \left(\frac{1 + \alpha_e^2}{1 + \sum \alpha_\beta^2}\right)^2 n_i f_{i0}\left(\frac{\Delta\omega}{\Delta k}\right) + \left(\frac{\alpha_i^2}{1 + \sum \alpha_\beta^2}\right)^2 n_e f_{e0}\left(\frac{\Delta\omega}{\Delta k}\right) \right\} \tag{65}$$

$$\text{with} \quad \alpha_\beta^2 = 1/(\Delta k)^2 \lambda_\beta^2, \quad \lambda_\beta^2 = v_\beta^2 / 2\,\omega_\beta^2. \tag{66}$$

Since for small arguments $f_{\alpha 0}(v)$ goes like v_α^{-1}, the first term of (65) dominates. Thus for arbitrary α_β the center of the spectrum has an essentially GAUSSIAN profile corresponding to the ion thermal velocity.

b) $|\Delta\omega|/\Delta k\, v_e \ll 1, \quad |\Delta\omega|/\Delta k\, v_i \gg 1.$

According to Eqs. (A 3), (A 4)

$$S(\Delta\omega, \boldsymbol{n}) = \frac{V}{\Delta k} \left(\left| \frac{1 + \alpha_e^2}{1 + \alpha_e^2 - (\omega_i/\Delta\omega)^2 \mu_i - i\,\Delta\gamma} \right|^2 n_i f_{i0}\left(\frac{\Delta\omega}{\Delta k}\right) + \left| \frac{(\omega_i/\Delta\omega)^2 \mu_i}{1 + \alpha_e^2 - (\omega_i/\Delta\omega)^2 \mu_i - i\,\Delta\gamma} \right|^2 n_e f_{e0}\left(\frac{\Delta\omega}{\Delta k}\right) \right) \tag{67}$$

with $$\mu_\alpha = 1 + \frac{3}{2} \left(\frac{\Delta k\, v_\alpha}{\Delta\omega}\right)^2, \quad \Delta\gamma = 2\sqrt{\pi} \sum_\alpha \left(\frac{\omega_\alpha}{\Delta\omega}\right)^2 \left(\frac{\Delta\omega}{\Delta k\, v_\alpha}\right)^3 \exp\left\{-\left(\frac{\Delta\omega}{\Delta k\, v_\alpha}\right)^2\right\}. \tag{68}$$

In this regime there is a more or less pronounced maximum near $(\Delta\omega)^2 = \omega_i^2 \mu_i (1 + \alpha_e^2)^{-1}$, corresponding to ion oscillations. $S(\Delta\omega, \boldsymbol{n})$ at this frequency is higher than the corresponding electron spectrum by a factor of $(1 + \alpha_e^2)^2/\alpha_e^4$.

c) $|\Delta\omega|/\Delta k\, v_\alpha \gg 1, \quad \alpha = e, i.$

The ion term can be neglected because of its exponential smallness. Thus

$$S(\Delta\omega, \boldsymbol{n}) = \frac{V}{\Delta k} \left| \frac{\omega_i^2 \mu_i}{(\Delta\omega)^2 - \omega_e^2 \mu_e - i(\Delta\omega)^2 \Delta\gamma} \right|^2 n_e f_{e0}\left(\frac{\Delta\omega}{\Delta k}\right). \tag{69}$$

The electron satellite line at the frequencies where the denominator is small is smaller by a factor of $(m_e/m_i)^2$ than in the electron spectrum and will hardly be visible.

7. Conclusions

In the foregoing sections, a microscopic model has been used to calculate scattered light the amplitude of which depends quadratically on the amplitude of the scattering light. It has been shown that by choosing two light frequencies appropriately resonances give higher intensity for the scattering process. A comparison of the results with those given in [2] is suggested. The effects in [2] seem to be different, however, since there a second order proces is found by combining a linear and a third-order amplitude to give a quadratic effect in the intensity.

Acknowledgements

The author wishes to thank Professor Dr. A. Schlüter for suggesting the subject of this paper and for helpful discussions.

This work has been undertaken as part of the joint research program of the Institut für Plasmaphysik and EURATOM.

Appendix A

The function
$$Z_\alpha\left(\frac{\omega}{k v_\alpha}\right) = \omega_\alpha^2 \int_{-\infty}^{+\infty} \frac{\mathrm{d}v\, f_{\alpha 0}(v)}{(\omega - k v)^2} = -\frac{1}{k^2 \lambda_\alpha^2}\left(1 + \frac{\omega}{k v_\alpha}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}v\, e^{-v^2}}{v - \omega/k v_\alpha}\right) \tag{A 1}$$
can be expressed in terms of known functions, see [13]
$$Z_\alpha(x) = -\frac{1}{k^2 \lambda_\alpha^2}\left(1 - i x \sqrt{\pi}\, e^{-x^2}[1 - \operatorname{erf}(i x)]\right). \tag{A 2}$$
For $\lim \operatorname{Im} \omega = -0$ the Taylor series or the asymptotic representation gives

for $|\omega|/k v_\alpha \gg 1$:
$$Z_\alpha\left(\frac{\omega}{k v_\alpha}\right) = -\frac{\omega_\alpha^2}{\omega^2}\left\{2 i \sqrt{\pi}\left(\frac{\omega}{k v_\alpha}\right)^3 \exp\left\{-\left(\frac{\omega}{k v_\alpha}\right)^2\right\} + 1 + \frac{3}{2}\left(\frac{k v_\alpha}{\omega}\right)^2 + \dots\right\}, \tag{A 3}$$
for $|\omega|/k v_\alpha \ll 1$:
$$Z_\alpha\left(\frac{\omega}{k v_\alpha}\right) = -\frac{1}{k^2 \lambda_\alpha^2}\left(1 - i\sqrt{\pi}\frac{\omega}{k v_\alpha} + \dots\right). \tag{A 4}$$
The corresponding formulas for $\varepsilon(\omega, k)$ can be found from
$$\varepsilon(\omega, k) = 1 - \sum Z_\alpha(\omega/k v_\alpha). \tag{A 5}$$

Appendix B

The integral
$$M_\alpha(\omega, \mathbf{r}) = \frac{1}{(2\pi)^3}\int \mathrm{d}^3k \exp\{-i\,\mathbf{k}\cdot\mathbf{r}\}\frac{1}{\alpha(\omega, k)}\frac{1}{k^2}\,\delta N_\alpha^{(0)}(\omega - \omega_s, \mathbf{k} - \mathbf{k}_s) \tag{B 1}$$
is easily calculated with the aid of (26), (2), and (6). For $\omega \neq \omega_s$
$$\begin{aligned} M_\alpha(\omega, \mathbf{r}) &= \sum_{i=1}^{N_\alpha}\int_0^\infty \mathrm{d}t \exp\{-i(\omega - \omega_s)\,t\}\frac{1}{(2\pi)^3}\int\frac{1}{k^2}\frac{1}{k^2 c^2 - \omega^2 + \omega_p^2}\exp\{i(\mathbf{k} - \mathbf{k}_s)\cdot\mathbf{r}_i(t)\}\exp\{-i\,\mathbf{k}\cdot\mathbf{r}\} \\ &= \sum_i \int_0^\infty \mathrm{d}t \exp\{-i(\omega - \omega_s)\,t\}\exp\{-i\,\mathbf{k}_s\cdot\mathbf{r}_i\}\frac{1}{(2\pi c)^2}\int_{-1}^{+1}\mathrm{d}\cos\vartheta\int_0^\infty \mathrm{d}k\,\frac{\exp\{-i\,k\,R_i^*\cos\vartheta\}}{k^2 - \tilde{k}^2} \\ &= \sum_i \int_0^\infty \mathrm{d}t \exp\{-i(\omega - \omega_s)\,t\}\exp\{-i\,\mathbf{k}_s\cdot\mathbf{r}_i\}\frac{1}{4\pi c^2}\frac{1}{k^2 R_i^*}\exp\{-i\,\tilde{k}\,R_i^*\} \end{aligned} \tag{B 2}$$
with
$$\tilde{k}\,c = (\omega^2 - \omega_p^2)^{1/2}\,\operatorname{sign}\operatorname{Re}\omega, \qquad R_i^* = |\mathbf{r} - \mathbf{r}_i(t)|. \tag{B 3}$$
In the wave zone there results from
$$|\mathbf{r} - \mathbf{r}_i| \approx r - \mathbf{n}\cdot\mathbf{r}_i, \qquad \mathbf{n} = \mathbf{r}/r, \qquad \mathbf{k} = k\,\mathbf{n}, \tag{B 4}$$
$$M_\alpha(\omega, \mathbf{r}) = \frac{1}{4\pi c^2 r}\exp\{-i\,\tilde{k}\,r\}\,\delta N_\alpha^{(0)}(\omega - \omega_s, \tilde{\mathbf{k}} - \mathbf{k}_s). \tag{B 5}$$

[13] B. Fried and S. Conte, The Plasma Dispersion Function, Academic Press, New York—London 1961.